THE CONSTRUCTION OF AN INTEGRAL-OPTIMAL PROGRAMMED CONTROL FUNCTION IN A LINEAR SYSTEM

(POSTROENIE PROGRAMMNOGO UPRAVLENIIA V LINEINOI SISTEME OPTIMAL'NOGO V INTEGRAL'NOM SMYSLE)

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The article describes a successive-approximations method for finding the optimum programmed control for a linear system and demonstrates the convergence of this method.

1. Statement of the problem. Let us consider the system of ordinary differential equations

$$X^{\cdot}(t) = A(t)\,X(t) + \sum_{i=1}^{r} B_i(t)\,u_i(t) + F(t) \tag{1.1}$$

with the initial condition

$$X(0) = X_0 \tag{1.2}$$

where $A(t)$ is an n-by-n matrix and $F(t)$, $B_i(t)$, $i = 1, \ldots, r$, are n-dimensional vectors. We shall assume that the elements of the matrix $A(t)$ and the components of the vectors $F(t)$, $B_i(t)$ ($i = 1, \ldots, r$) are continuous real functions, specified on $[0, T]$. Let T be fixed.

We shall also assume that the control functions $u_1(t), \ldots, u_r(t)$ are real functions of time specified on $[0, T]$ and satisfy the inequalities

$$|u_i(t)| \leqslant 1 \qquad (i = 1, \ldots, r) \tag{1.3}$$

The entire class of such functions (we shall denote it by U) may be assumed to coincide with the totality of piecewise continuous functions or, in the general case, with the totality of measurable functions.

Let $X(t, u)$ be a solution of equation (1.1) with the initial conditions (1.2) for a specified $u(t) = u_1(t), \ldots, u_r(t)$, let $N(t)$ be a symmetric negative-definite square matrix with continuous coefficients, and let * denote transposition.

It is required to find $u(t) \in U$ such that the functional

$$J(u) = \int_0^T X^*(t, u)\, N(t)\, X(t, u)\, dt \tag{1.4}$$

takes on its minimum possible value.

We shall describe below a successive-approximations method for minimising the functional equation (1.4). The method is illustrated for the case in which the functional is of the form

$$J(u) = \int_0^T X^*(t, u)\, X(t, u)\, dt \tag{1.5}$$

2. Solution of the auxiliary problem. Let $C(t)$ be an arbitrary continuous real n-dimensional vector function; it is required that we find among the control functions mentioned in Section 1, the one which will minimize the functional

$$J_C(u) = \int_0^T X^*(t, u)\, C(t)\, dt \tag{2.1}$$

If we know a fundamental system $Y(t)$ for the homogeneous part of equation (1.1), the solution of the system (1.1) with the initial conditions (1.2) is obtained by Cauchy's formula

$$X(t) = Y(t)\, X_0 + \int_0^t \sum_{i=1}^r Y(t)\, Y^{-1}(\tau)\, B_i(\tau)\, u_i(\tau)\, d\tau + \int_0^t Y(t)\, Y^{-1}(\tau)\, F(\tau)\, d\tau \tag{2.2}$$

This means that for any choice of $u_1(t), \ldots, u_r(t)$ the value of the functional (2.1) can be calculated

$$J_C(u) = \int_0^T C^*(t) \Big[Y(t)\, X_0 + \int_0^t Y(t)\, Y^{-1}(\tau)\, F(\tau)\, d\tau + \int_0^t \sum_{i=1}^r Y(t)\, Y^{-1}(\tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \Big] dt$$

Let us consider

$$J_i = \int_0^T \int_0^t C^*(t)\, [Y(t)\, Y^{-1}(\tau)\, B_i(\tau)]\, u_i(\tau)\, d\tau dt \qquad (i = 1, \ldots, r)$$

We integrate by parts

$$J_i = \int_0^T [\omega(T) - \omega(t)]^* [Y^{-1}(t) B_i(t)] u_i(t)\, dt$$

$$\omega(t) = \int_0^t Y^*(\tau) c(\tau)\, d\tau, \qquad \omega(T) - \omega(t) = \int_t^T Y^*(\tau) C(\tau)\, d\tau$$

Since

$$J_C(u) = \int_0^T C^*(t) Y(t) X_0 dt + \sum_{i=1}^r J_i + \int_0^T C^*(t) \int_0^t Y(t) Y^{-1}(\tau) F(\tau)\, d\tau\, dt$$

it follows that $J_C(u)$ will take on a minimum value if and only if

$$u_i(t) = -\operatorname{sign}\left\{\left[\int_t^T Y^*(\tau) C(\tau)\, d\tau\right]^* [Y^{-1}(t) B_i(t)]\right\} \qquad (i = 1, \ldots, r) \quad (2.3)$$

where

$$\operatorname{sign} a = 1 \quad (a > 0), \qquad \operatorname{sign} a = 0 \quad (a = 0), \qquad \operatorname{sign} a = -1 \quad (a < 0)$$

Thus, if the fundamental matrix $Y(t)$ is known, the solution of the auxiliary problem is given by the formula (2.3).

3. The successive-approximation method. Let us take an arbitrary control function $u^1(t) \in U$. Let $X_1(t)$ be a solution of the system (1.1) with the initial condition (1.2) for $u(t) = u^1(t)$. We solve the auxiliary problem for $C(t) = X_1(t)$. Let $v^1(t)$ be the control function which yields a minimum value for the auxiliary problem, and let $\chi_1(t)$ be the solution corresponding to this control function. We set

$$\int_0^T X_n^*(t) X_n(t)\, dt = \theta_n, \qquad \int_0^T X_n^*(t) \chi_n(t)\, dt = \varphi_n$$

$$\int_0^T \chi_n^*(t) \chi_n(t)\, dt = \Psi_n, \qquad \int_0^T X^*(t) X(t)\, dt = \theta$$

Then $\varphi_1 \leqslant \theta_1$. If the equality $\varphi_1 = \theta_1$ holds, then $u^1(t)$ is already the optimal control function and the process is terminated. In the case $\varphi_1 < \theta_1$ we proceed as follows

a) if $\psi_1 \leqslant \varphi_1$, we set

$$u^2(t) = v^1(t), \qquad X_2(t) = \chi_1(t)$$

b) if $\psi_1 > \varphi_1$, we construct

$$u(t) = \alpha u^1(t) + (1 - \alpha) v^1(t) \qquad (0 < \alpha < 1)$$

In view of the linearity of the system (1.1), the corresponding solution is of the form

$$X(t) = \alpha X_1(t) + (1-\alpha)\chi_1(t)$$

We select α such that θ takes on its minimum value (among all $\alpha \in (0, 1)$). As is readily shown, this will be true for

$$\alpha = \frac{\psi_1 - \varphi_1}{\theta_1 - 2\varphi_1 + \psi_1} \tag{3.1}$$

We take

$$u^2(t) = \alpha u^1(t) + (1-\alpha) v^1(t), \qquad X_2(t) = \alpha X_1(t) + (1-\alpha)\chi_1(t)$$

where α is chosen by formula (3.1); then $\theta_2 < \theta_1$, and so on.

Thus, we can always select $X_2(t)$ if $u^1(t)$ is not optimal. Solving the auxiliary problem for $C(t) = X_2(t)$, we obtain $\chi_2(t)$ and proceed thereafter as before. The resulting sequences $u^1(t)$, $u^2(t)$, ..., $X_1(t)$, $X_2(t)$, ..., are such that $\theta_{m+1} \leqslant \theta_m$; if the equality holds in any of these cases, the optimal control function has been found and the process is terminated. The sequence θ_m is convergent. If Q^2 denotes the minimum possible value of the functional (1.5) – the lower bound of the functional (1.5) on $u(t) \in U$ is actually reached, as will be shown later – then

$$\sigma_m \frac{\varphi_m^2}{\theta_m} \leqslant Q^2 \leqslant \theta_m, \qquad \sigma_m = 1 \quad (\varphi_m > 0), \qquad \sigma_m = 0 \quad (\varphi_m \leqslant 0)$$

Thus, at each step we know the intervals in which the minimum of the functional occurs.

4. The fundamental theorem. The successive-approximations described in Section 3 monotonically reduce the values of the functional $J(u)$.

The sequence of functions $X_1(t)$, $X_2(t)$, ..., converges on $[0, T]$ to a continuous vector function $C_0(t)$, and the sequence of values θ_1, θ_2, ... converges to

$$\min_{u\in U} \int_0^T X^*(t, u)\, X(t, u)\, dt$$

An optimal control function exists. Moreover, for those i for which the set Ω_i of the zeros of the function

$$\Phi_i(t) = \left[\int_t^T Y^*(\tau)\, C_0(\tau)\, d\tau\right]^* [Y^{-1}(t)\, B_i(t)]$$

is of the measure zero, the corresponding $u_i^m(t)$ converge in measure to a uniquely defined optimal control function $u_i(t)$.

Proof. It has already been established that

$$\| X_m \| = \sqrt{\theta_m} \to C \quad \text{if} \quad m \to \infty$$

If $X_0 \neq 0$, the $C > 0$. We first show that

$$\lim_{m\to\infty} \left[\min_{u \in U} \int_0^T X^* (t, u)\, X_m (t)\, dt \right] = \inf \lim_{m\to\infty} \varphi_m \geqslant C^2 \tag{4.1}$$

Let us assume that this is false, that is, that there exists an $r_0 > 0$, such that for every N we can find an $m > N$ for which

$$\varphi_m = C^2 - r_m \leqslant C^2 - r_0 \tag{4.2}$$

We shall consider only those $X_m(t)$ for which the inequality (4.2) holds. It should be noted in that case $\psi_m > \varphi_m$, since otherwise we would have $\psi_m \leqslant \theta_m$ by our rule for the selection of $X_{m+1}(t) = X_m(t)$; but for $\chi_m(t)$

$$\psi_m \leqslant \varphi_m \leqslant C^2 - r_0 < C^2$$

which contradicts the statement that $\| X_m \|$ is a decreasing sequence with the limit C. Therefore $\psi_m > \varphi_m$, so that

$$X_{m+1} (t) = \alpha X_m (t) + (1 - \alpha)\, \chi_m (t)$$

where α is chosen by formula (3.1). Then

$$\theta_{m+1} = \alpha^2 [\theta_m - 2\varphi_m + \psi_m] + 2\alpha [\varphi_m - \psi_m] + \psi_m = \frac{\theta_m \psi_m - \varphi_m^2}{\theta_m - 2\varphi_m + \psi_m}$$

Having selected m such that

$$\theta_m = C^2 + \varepsilon_m, \qquad \varepsilon_m \to 0 \qquad \text{if} \quad m \to \infty$$

we have

$$\theta_{m+1} = \frac{C^2 [\psi_m + \psi_m \varepsilon_m / C^2 - C^2 + 2r_m - r_m^2 / C^2]}{\psi_m + \varepsilon_m - C^2 + 2r_m} < C^2$$

since ψ_m is bounded (by the properties of a linear system, each coordinate is bounded, so that the integral ψ_m is also bounded). But the inequality $\theta_{m+1} < C^2$ is impossible, since $\| X_m \|$ tends to C from above. From this follows the inequality (4.1). But, since $\varphi_m \leqslant \theta_m$, it is true that $\varphi_m \to C^2$ as $m \to \infty$.

Let us now consider the sequence

$$X_1 (t),\ X_2 (t),\ \ldots \tag{4.3}$$

This is a sequence of vector functions which are differentiable on $[0, T]$. From the properties of linear systems it follows that the sequences of coordinates

$$(x_1^1(t), x_2^1(t), \ldots), \quad (x_1^2(t), x_2^2(t), \ldots), \quad \ldots, \quad (x_1^n(t), x_2^n(t), \ldots)$$

are equicontinuous and uniformly bounded on $[0, T]$.

Then, by the Arzela-Ascoli theorem [1], we can select a convergent sub-sequence of vector functions such that their limit is also a continuous vector function. Let us assume that there are two sub-sequences

$$X_{m_1}(t), \quad X_{m_2}(t), \ldots, \qquad X_{m_j}(t) \xrightarrow[j\to\infty]{} C_0(t)$$
$$X_{m_1}'(t), \quad X_{m_2}'(t), \ldots, \qquad X_{m_i}'(t) \xrightarrow[i\to\infty]{} C_0'(t)$$

and let

$$C_0(t) \neq C_0'(t), \qquad \|C_0\| = \|C_0'\| = C$$

Then

$$\int_0^T C_0^*(t)\, C_0'(t)\, dt \leqslant C^2 - \rho_0, \qquad \rho_0 > 0 \qquad \int_0^T [C_0(t) - C_0'(t)]^2\, dt > 0$$

(the last inequality holds by virtue of the fact that $C_0(t)$ and $C_0'(t)$ are continuous vector functions and do not coincide identically). Then

$$\int_0^T C_0^*(t)\, C_0'(t)\, dt < \frac{1}{2}\left[\int_0^T C_0^2(t)\, dt + \int_0^T C_0'^2(t)\, dt\right] = C^2$$

But, having selected $X_{m_j}(t)$ close to $C_0(t)$ and X_{m_i}' close to $C_0'(t)$, we obtain

$$\int_0^T X_{m_i}'^*(t)\, X_{m_j}(t)\, dt \leqslant C^2 - \rho_0 + \varepsilon < C^2$$

since ε can be made arbitrarily small. But in that case

$$\min_{u\in U}\int_0^T X_{m_j}^*(t)\, X(t, u)\, dt \leqslant \int_0^T X_{m_j}^*(t)\, X_{m_i}'(t)\, dt < C^2$$

which contradicts the inequality (4.1). Hence the sequence of functions (4.3) converges to some continuous vector function $C_0(t)$.

Let us show that

$$\theta_m \to \min_{u\in U} \int_0^T X^* (t, u)\, X (t, u)\, dt \quad \text{as} \quad m \to \infty$$

Suppose this is false: that is, suppose that there exists some $w(t) \in U$ such that

$$\int_0^T X^* (t, w)\, X (t, w)\, dt < C^2$$

Then, having selected $X_m(t)$ close to $C_0(t)$, we obtain

$$\int_0^T X_m^* (t)\, X(t, w)\, dt < C^2$$

Then we also have

$$\min_{u\in U} \int_0^T X_m^* (t)\, X (t, u)\, dt \leqslant \int_0^T X_m^* (t)\, X (t, w)\, dt < C^2$$

which contradicts the inequality (4.1). Thus

$$\theta_m \to \min_{u\in U} \int_0^T X^* (t, u)\, X (t, u)\, dt \quad \text{as} \quad m \to \infty$$

but in that case we also have

$$\int_0^T X_m^* (t)\, C_0 (t)\, dt \to \min_{u\in U} \int_0^T X^* (t, u)\, C_0 (t)\, dt \quad \text{as} \quad m \to \infty$$

and, by Section 2, for those i for which the set Ω_i of the zeros of the functions

$$\Phi_i (t) = \left[\int_0^T Y^* (\tau)\, C_0 (r)\, d\tau \right]^* [Y^{-1} (t)\, B_i (t)]$$

is of measure zero, the corresponding $u_i^m(t)$ converge in measure to a uniquely defined optimal control function

$$u_i^* (t) = - \operatorname{sign} \Phi_i (t) \tag{4.4}$$

We have thus proved that such an optimal control function exists.

In [2, pp. 146, 147] it is, in fact, proved that in an arbitrary case (independently of the measure of the sets Ω_i) an optimal control function exists. On the sets $[0, T] - \Omega_i$ the corresponding optimal control function $u_i(t)$ is uniquely defined and coincides with (4.4).

The optimal solution (trajectory) $X(t)$ is uniquely defined, as has already been shown. The vector function $X(t)$ has a uniquely defined derivative, but this is not enough to prove the uniqueness of the optimal control function, since the equation

$$\sum_{i=1}^{r} B_i(t)\, u_i(t) = X^\circ(t) - A(t)\, X(t) - F(t) \qquad (t \in [0, T])$$

may be satisfied by several control functions $u(t) \in U$ (here $X(t)$ is assumed to be a known vector function).

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